

# Unconditionally secure quantum coin flipping

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Quantum coin flipping (QCF) is an essential primitive for quantum cryptography. Unconditionally secure strong QCF with an arbitrarily small bias was widely believed to be impossible. But basing on a problem which cannot be solved without quantum algorithm, here we propose such a QCF protocol, and show how it manages to evade all existing no-go proofs on QCF.

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## I. INTRODUCTION

Quantum coin flipping (QCF) [1], a.k.a. quantum coin tossing, is aimed to provide a method for two separated parties Alice and Bob to generate a random bit value  $c = 0$  or  $1$  remotely, while they do not trust each other. If the parties have opposite desired values, e.g., Alice wants  $c = 0$  while Bob wants  $c = 1$ , then it is called weak QCF. Or if their desired values are random, then it is called strong QCF. Here we concentrate on strong QCF only. Such a QCF protocol is considered secure if neither party can bias the outcome, so that  $c = 0$  and  $c = 1$  will both occur with the equal probability  $1/2$ , just as if they are tossing an ideal fair coin.

QCF is an essential element of cheat-sensitive protocols [2–4]. It is also closely related with quantum bit commitment (QBC) and quantum oblivious transfer [5], which are the building blocks for more complicated quantum multi-party secure computation protocols [6]. However, it is widely believed that unconditionally secure QCF is impossible. More rigorously, let  $\varepsilon$  denote the bias of a QCF protocol, such that the dishonest party can force the outcome  $c = 0$  or  $c = 1$  to occur with probability  $1/2 + \varepsilon > 1/2$ . Unconditional security requires that  $\varepsilon$  should be able to be made arbitrarily small by increasing some parameters in the protocol, without relying on any computational assumption or experimental constraint. But there were proofs claiming that this can never be achieved. Instead, a lower bound of the bias exists, which is  $\varepsilon \geq 1/\sqrt{2} - 1/2$  [5], [7–19]. These no-go proofs are considered as casting very serious doubt on the security of quantum cryptography in the so-called “post-cold-war” applications [7]. All previously proposed QCF protocols (e.g., Refs. [20–26] and the references therein) are, unfortunately, limited by this bound.

Nevertheless, we find that this negative result is not sufficiently general to cover all QCF protocols. Here we will propose an unconditionally secure QCF protocol, and show one-by-one why the no-go proofs [5], [7–19] fail to apply to our protocol.

## II. THE LIE-DETECTING PROBLEM

Since it is an important theoretical problem whether unconditionally secure QCF exists in principle, here for simplicity, we only consider the ideal case without practical imperfections, such as transmission errors, detection loss or dark counts, etc.

Let us denote the state of a qubit as  $|p, q\rangle$ , where  $p = 0, 1$  indicates the basis while  $q = 0, 1$  distinguishes the two orthogonal states in the same basis. That is,  $|0, 0\rangle$  and  $|0, 1\rangle$  are the eigenstates of the  $p = 0$  basis, while  $|1, 0\rangle \equiv (|0, 0\rangle + |0, 1\rangle)/\sqrt{2}$  and  $|1, 1\rangle \equiv (|0, 0\rangle - |0, 1\rangle)/\sqrt{2}$  are the eigenstates of the  $p = 1$  basis. Our protocol is built around the solutions to the following problem.

*Lie-detecting Problem:*

Alice sends Bob  $s$  qubits  $\beta_i$  ( $i \in S \equiv \{1, \dots, s\}$ ). Then Bob announces his “fake” measurement result  $|p'_i, q'_i\rangle_\beta \in \{|0, 0\rangle, |1, 0\rangle, |0, 1\rangle, |1, 1\rangle\}$  for each  $\beta_i$ , which is allowed to be different from his actual measurement result  $|p'_i, q'_i\rangle_\beta$ . That is, it can be either of the following three types of lies:

$$\begin{aligned} \text{type } a \text{ lies} : & p''_i = p'_i \wedge q''_i = \neg q'_i; \\ \text{type } b \text{ lies} : & p''_i = \neg p'_i \wedge q''_i = q'_i; \\ \text{type } c \text{ lies} : & p''_i = \neg p'_i \wedge q''_i = \neg q'_i. \end{aligned} \quad (1)$$

Now the question is: how many lies can Alice detect? More precisely, suppose that types  $a$ ,  $b$  and  $c$  lies occur with frequencies  $f_a$ ,  $f_b$  and  $f_c$ , respectively. Our task is to express the total number of Alice’s detected lies with the parameters  $f_a$ ,  $f_b$ ,  $f_c$  and  $s$ .

Note that it does not matter Bob measures each qubit before or after he announces  $|p'_i, q'_i\rangle_\beta$ . Even if he announced without measuring it, he can still measure it later and obtain  $|p'_i, q'_i\rangle_\beta$ , then compares it with  $|p''_i, q''_i\rangle_\beta$  to learn the type of lies that it belongs to. In this case, if  $p'_i$  is randomly chosen, then the values of  $p''_i$  and  $q''_i$  do not have a fixed relationship with  $p'_i$  and  $q'_i$ , which is equivalent to announcing a type  $a$ ,  $b$  or  $c$  lie or a honest result with the equal probability  $1/4$ . Else if Bob chooses  $p'_i = p''_i$  ( $p'_i \neq p''_i$ ), then the type  $a$  lie and the honest result (the type  $b$  and  $c$  lies) occur with the equal probability  $1/2$ . Either way, Bob is still able to control the values of  $f_a$ ,  $f_b$  and  $f_c$ .

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Meanwhile, from Alice's point of view, some lies can be identified even though Bob has not measured. For example, if Bob announces  $|p_i'', q_i''\rangle_\beta = |0, 0\rangle_\beta$  while Alice knows that she actually sent  $|0, 1\rangle_\beta$ , she knows for sure that this  $|p_i'', q_i''\rangle_\beta$  must be a lie no matter Bob has performed the measurement or not. When Bob has not measured the corresponding  $\beta_i$ , this result means that even if he later decides to measure it in the basis  $p_i'' = 0$  honestly, he can only obtain  $|p_i', q_i'\rangle_\beta = |0, 1\rangle_\beta$  so that his announced  $|p_i'', q_i''\rangle_\beta$  ends up as a type *a* lie. Else if he uses  $p_i'' = 1$  as the measurement basis, then  $|p_i'', q_i''\rangle_\beta$  will inevitably become a type *b* or *c* lie. In any case, he can never obtain the result  $|p_i', q_i'\rangle_\beta = |0, 0\rangle_\beta$  so that he cannot make  $|p_i'', q_i''\rangle_\beta$  an honest announcement. Therefore, Alice can detect lies as usual even if Bob delays his measurement.

### A. Algorithm I: the “semi-classical” algorithm

A very intuitive and ordinary solution of the problem is as follows.

*The state:* Alice determines the values of  $p_i$  and  $q_i$  ( $i \in S$ ) beforehand, and prepares the initial state of each  $\beta_i$  as a pure state  $|p_i, q_i\rangle_\beta$  non-entangled with any other system.

*Lie-detecting strategy:* After Bob announced  $|p_i'', q_i''\rangle_\beta$ , Alice compares it with  $|p_i, q_i\rangle_\beta$ . Whenever she finds  $p_i'' = p_i$  while  $q_i'' = \neg q_i$ , she knows that Bob told a lie. This is because in the ideal setting, once Bob chose the basis  $p_i'$  correctly (i.e., it matches Alice's  $p_i$ ), he should never find a wrong  $q_i'$  value in his measurement. Thus if he announces  $|p_i', q_i'\rangle_\beta$  as  $|p_i'', q_i''\rangle_\beta$  honestly, there should not be  $q_i'' = \neg q_i$  while  $p_i'' = p_i$ . On the other hand, if  $p_i'' = \neg p_i$  Alice will not be able to judge Bob's announcement. This is because when Bob chose the wrong basis to measure the qubit, any result is possible due to quantum uncertainty.

*Result:* The total number of lies that Alice detected with this algorithm can be evaluated as follows. Whenever Bob did not lie, Alice has no way to claim that he did. So we need to concentrate only on the qubits for which Bob indeed lies. Since Bob chooses the actual measurement basis  $p_i'$  randomly, for about  $s/2$  qubits Bob will choose the correct basis  $p_i' = p_i$  by chance. Among these qubits, all type *a* lies will inevitably be detected by Alice, while no types *b* and *c* can be judged for the reason stated above. Therefore Alice finds about  $f_a s/2$  lies. Meanwhile for the rest  $s/2$  qubits which Bob measured with the wrong basis  $p_i' = \neg p_i$ , the probabilities of finding  $q_i' = q_i$  or  $q_i' = \neg q_i$  are both  $1/2$ . Therefore when Bob tells a type *b* or *c* lie, the announced basis  $p_i'' = \neg p_i'$  becomes the correct basis so that the condition  $p_i'' = p_i$  is satisfied, and  $q_i'' = \neg q_i$  occurs with probability  $1/2$ . Thus Alice finds about  $(f_b + f_c)(1/2)(s/2)$  lies. But no lie *a* will be detected in this case since announcing  $p_i'' = p_i'$  will then be recognized by Alice as the wrong basis. All

in all, the total number of lies detected by Alice is about

$$l \sim (\frac{1}{2}f_a + \frac{1}{4}f_b + \frac{1}{4}f_c)s. \quad (2)$$

Note that Alice needs not to know the values of  $f_a$ ,  $f_b$  and  $f_c$ . She simply follows the above lie-detecting strategy and this number can automatically be reached.

This algorithm is called “semi-classical” because the quantum property of the qubits is not fully utilized, except for the quantum uncertainty involved in the measurement. Now we shall see that if Alice further makes use of quantum entanglement, the problem can be solved more efficiently.

Suppose that at the beginning, Alice introduces an ancillary system  $\alpha_i$  entangled with each  $\beta_i$ . She sends Bob  $\beta_i$  and keeps  $\alpha_i$  at her side. Since Bob's actual result  $|p_i', q_i'\rangle_\beta$  has four possible values  $|0, 0\rangle$ ,  $|0, 1\rangle$ ,  $|1, 0\rangle$  and  $|1, 1\rangle$ , the general form of the state of  $\alpha_i \otimes \beta_i$  can be taken as

$$\begin{aligned} |\alpha_i \otimes \beta_i\rangle = & c_x |x\rangle_\alpha \otimes |0, 0\rangle_\beta + c_y |y\rangle_\alpha \otimes |1, 0\rangle_\beta \\ & + c_{x'} |x'\rangle_\alpha \otimes |0, 1\rangle_\beta + c_{y'} |y'\rangle_\alpha \otimes |1, 1\rangle_\beta. \end{aligned} \quad (3)$$

By choosing different combinations of the superposition coefficients  $c_x$ ,  $c_y$ ,  $c_{x'}$  and  $c_{y'}$ , we find that the following three algorithms are especially useful for our current purpose.

### B. Algorithm II: quantum algorithm with maximally entangled states

*The state:* Alice takes

$$|\alpha_i \otimes \beta_i\rangle = (|x\rangle_\alpha \otimes |0, 0\rangle_\beta + |y\rangle_\alpha \otimes |0, 1\rangle_\beta) / \sqrt{2}, \quad (4)$$

which is the maximally entangled state of  $\alpha_i \otimes \beta_i$ . Here and in the following context  $|x\rangle_\alpha$  and  $|y\rangle_\alpha$  are taken to be orthogonal to each other.

*Lie-detecting strategy:* After Bob announced  $|p_i'', q_i''\rangle_\beta$ , Alice measures  $\alpha_i$  in the basis which forces  $\beta_i$  to collapse to  $p_i''$ . (Note that the measurement of Alice and Bob on the entangled state is permutable. Therefore, though the actual case could be that Bob has measured  $\beta_i$  and caused  $\alpha_i$  to collapse before Alice performs her measurement, it is equivalent to the case where Alice has measured  $\alpha_i$  and caused  $\beta_i$  to collapse first.) That is, if  $p_i'' = 0$ , she measures  $\alpha_i$  in the basis  $\{|x\rangle_\alpha, |y\rangle_\alpha\}$  so that  $\beta_i$  collapses to  $|0, 0\rangle_\beta$  or  $|0, 1\rangle_\beta$ . Else if  $p_i'' = 1$ , she measures  $\alpha_i$  in the basis  $\{(|x\rangle_\alpha + |y\rangle_\alpha)/\sqrt{2}, (|x\rangle_\alpha - |y\rangle_\alpha)/\sqrt{2}\}$  so that  $\beta_i$  collapses to  $|1, 0\rangle_\beta$  or  $|1, 1\rangle_\beta$ . As a result, the real “initial” basis  $p_i$  of the qubit  $\beta_i$  always equals to Bob's announced basis  $p_i''$ . Then Alice knows that Bob told a lie whenever she finds  $q_i'' = \neg q_i$ .

*Result:* Since  $p_i$  always equals to  $p_i''$  in this algorithm, it can be seen that the probability for any type of lies to be detected by Alice is doubled when comparing with

that of the “semi-classical” algorithm, where  $p_i = p''_i$  occurs at half of the cases only. Therefore, if Alice measures all  $\alpha_i$ 's in the  $s$  pairs of entangled states, the total number of lies that she can detect is  $2l \sim (f_a + f_b/2 + f_c/2)s$ . On the other hand, if she is required to detect  $l$  (defined by Eq. (2)) lies only, just as what can be achieved in the “semi-classical” algorithm, the number of  $\alpha_i$  which she needs to measure is  $s/2$  only. The other  $s/2$  pairs of entangled states can be left intact.

In fact, the “semi-classical” algorithm can be written in an equivalent form, in which the initial state is also prepared as Eq. (4), but Alice always picks a random basis  $p_i$  for measuring  $\alpha_i$ , instead of choosing  $p_i$  according to Bob's  $p''_i$ . Then to detect  $l$  lies, she has to measure all the  $s$  entangled states. In this sense, the current quantum algorithm is more efficient than the “semi-classical” one as less  $\alpha_i$  is measured.

### C. Algorithm III: quantum algorithm with non-maximally entangled states

Although the above algorithm is the best we found for Alice to detect as much lies as possible, it is not the most efficient one if only  $l$  lies are required to be detected. With other forms of  $|\alpha_i \otimes \beta_i\rangle$ , the number of  $\alpha_i$  that Alice needs to measure can be further reduced.

*The state:* Alice prepares

$$|\alpha_i \otimes \beta_i\rangle = \cos \theta_i |x\rangle_\alpha \otimes |0, q_i\rangle_\beta + \sin \theta_i |y\rangle_\alpha \otimes |1, q_i\rangle_\beta, \quad (5)$$

where  $q_i \in \{0, 1\}$  and  $0 < \theta_i < \pi/2$ . Note that  $\theta_i$  needs not to be different for each  $i$ . For example, Alice can always take  $\theta_i = \pi/4$ . But to make our analysis sufficiently general to cover all algorithms with the same property, in this subsection we do not limit  $\theta_i$  to a fixed value.

*Lie-detecting strategy:* After Bob announced  $|p''_i, q''_i\rangle_\beta$ , Alice divides the  $s$  pairs of entangled states into two subsets: Set  $M$  includes all those satisfying  $q''_i \neq q_i$ , and set  $U$  includes all those satisfying  $q''_i = q_i$ . When Alice's goal is to detect  $l$  lies only, she can simply leave all the qubits in  $U$  unmeasured. Meanwhile, for these in  $M$  she measures each  $\alpha_i$  in the basis  $\{|x\rangle_\alpha, |y\rangle_\alpha\}$ . She sets  $p_i = 0$  ( $p_i = 1$ ) if the measurement result is  $|x\rangle_\alpha$  ( $|y\rangle_\alpha$ ). Then she sets  $L = \{i \in M | p_i = p''_i\}$  as the set of qubits for which she detected lies.

*Result:* After applying this strategy to all  $\alpha_i \otimes \beta_i$  ( $i \in S$ ), set  $S$  is divided into three subsets:

$U \equiv \{i \in S | q''_i = q_i\}$ , for which Alice has not measured  $\alpha_i$ ,

$L \equiv \{i \in S | (q''_i \neq q_i) \wedge (p''_i = p_i)\}$ , for which Alice has measured  $\alpha_i$  and detected that  $|p''_i, q''_i\rangle_\beta$  is a lie, and

$N \equiv \{i \in S | (q''_i \neq q_i) \wedge (p''_i \neq p_i)\}$ , for which Alice has measured  $\alpha_i$  and she cannot judge whether  $|p''_i, q''_i\rangle_\beta$  is honest or a lie.

In addition,  $M = L \cup N = \{i \in S | q''_i \neq q_i\}$ .

Here we will show that set  $N$  has a distinguishing property, that it does not contain any type  $b$  lie. Suppose that

Alice takes  $q_i = 0$ . In this case Eq. (5) becomes

$$|\alpha_i \otimes \beta_i\rangle = \cos \theta_i |x\rangle_\alpha \otimes |0, 0\rangle_\beta + \sin \theta_i |y\rangle_\alpha \otimes |1, 0\rangle_\beta, \quad (6)$$

which can also be written as

$$\begin{aligned} |\alpha_i \otimes \beta_i\rangle &= (\cos \theta_i |x\rangle_\alpha + \frac{\sin \theta_i}{\sqrt{2}} |y\rangle_\alpha) \otimes |0, 0\rangle_\beta \\ &\quad + \frac{\sin \theta_i}{\sqrt{2}} |y\rangle_\alpha \otimes |0, 1\rangle_\beta, \end{aligned} \quad (7)$$

or

$$\begin{aligned} |\alpha_i \otimes \beta_i\rangle &= (\frac{\cos \theta_i}{\sqrt{2}} |x\rangle_\alpha + \sin \theta_i |y\rangle_\alpha) \otimes |1, 0\rangle_\beta \\ &\quad + \frac{\cos \theta_i}{\sqrt{2}} |x\rangle_\alpha \otimes |1, 1\rangle_\beta. \end{aligned} \quad (8)$$

According to the lie-detecting strategy, Alice will measure  $\alpha_i$  in the basis  $\{|x\rangle_\alpha, |y\rangle_\alpha\}$  only if Bob announces  $q''_i = 1$ . In this case, if  $p''_i = 0$ , i.e., Bob announces  $|p''_i, q''_i\rangle_\beta = |0, 1\rangle_\beta$ , Eq. (7) shows that when this is the honest result, i.e., Bob indeed measured  $\beta_i$  in the  $p'_i = 0$  basis and finds  $|p'_i, q'_i\rangle_\beta = |0, 1\rangle_\beta$ , then  $\alpha_i$  should collapse to  $|y\rangle_\alpha$ . Therefore, if Alice's measurement result on  $\alpha_i$  is  $|x\rangle_\alpha$ , she knows for sure that  $|p''_i, q''_i\rangle_\beta = |0, 1\rangle_\beta$  is a lie. Especially, if this  $|p''_i, q''_i\rangle_\beta$  is a type  $b$  lie, i.e., Bob actually measured  $\beta_i$  in the  $p'_i = 1$  basis and the actual result is  $|p'_i, q'_i\rangle_\beta = |1, 1\rangle_\beta$ , Eq. (8) shows that  $\alpha_i$  will collapse to  $|x\rangle_\alpha$  with probability 1 so that Alice can always detect it and include it in set  $L$ . Consequently, if Alice measures  $\alpha_i$  and finds that the result is  $|y\rangle_\alpha$ , she knows with certainty that  $|p''_i, q''_i\rangle_\beta = |0, 1\rangle_\beta$  is not a type  $b$  lie.

On the other hand, if Bob announces  $|p''_i, q''_i\rangle_\beta = |1, 1\rangle_\beta$ , Eq. (8) shows that when  $|p''_i, q''_i\rangle_\beta$  is the honest result, then  $\alpha_i$  should collapse to  $|x\rangle_\alpha$ . Therefore, if Alice measures  $\alpha_i$  and finds that the result is  $|y\rangle_\alpha$ , she knows for sure that  $|p''_i, q''_i\rangle_\beta = |1, 1\rangle_\beta$  is a lie. Again, if this  $|p''_i, q''_i\rangle_\beta$  is a type  $b$  lie, i.e., Bob actually measured  $\beta_i$  in the  $p'_i = 0$  basis and the actual result is  $|p'_i, q'_i\rangle_\beta = |0, 1\rangle_\beta$ , Eq. (7) shows that  $\alpha_i$  will collapse to  $|y\rangle_\alpha$  with probability 1 so that Alice can always detect it and include it in set  $L$ . Consequently, if Alice measures  $\alpha_i$  and finds that the result is  $|x\rangle_\alpha$ , she knows with certainty that  $|p''_i, q''_i\rangle_\beta = |1, 1\rangle_\beta$  is not a type  $b$  lie.

The case where Alice takes  $q_i = 1$  can also be analyzed similarly. Once again, if Bob announces  $|p''_i, q''_i\rangle_\beta = |0, 0\rangle_\beta$  ( $|p''_i, q''_i\rangle_\beta = |1, 0\rangle_\beta$ ) and Alice's measurement result on  $\alpha_i$  is  $|y\rangle_\alpha$  ( $|x\rangle_\alpha$ ), she knows with certainty that  $|p''_i, q''_i\rangle_\beta$  is not a type  $b$  lie. Otherwise, she knows that it must be a lie. Namely, all type  $b$  lies in set  $M$  will be detected and put into set  $L$ , so that set  $N$  will contain no type  $b$  lie at all.

The sizes of sets  $U$ ,  $L$ ,  $M$  and  $N$  can then be evaluated as follows. Denote the frequency that Bob announces the honest results as

$$f_h \equiv 1 - f_a - f_b - f_c. \quad (9)$$

Eq. (7) shows that if Bob measures  $\beta_i$  in the  $p'_i = 0$  basis, the result  $q'_i \neq q_i$  will occur with probability  $(\sin^2 \theta_i)/2$ , while Eq. (8) shows that if Bob measures  $\beta_i$  in the  $p'_i = 1$  basis, the result  $q'_i \neq q_i$  will occur with probability  $(\cos^2 \theta_i)/2$ . Thus the average probability for him to find  $q'_i \neq q_i$  is  $((\sin^2 \theta_i)/2 + (\cos^2 \theta_i)/2)/2 = 1/4$ . For types  $a$  and  $c$  lies, Bob always announces  $q''_i \neq q'_i$ . Thus they will satisfy  $q''_i = q_i$  and fall into set  $U$  with probability  $1/4$ . For honest results and type  $b$  lies, Bob always announces  $q''_i = q'_i$ . Thus they will satisfy  $q''_i = q_i$  and fall into set  $U$  with probability  $3/4$ . Therefore, the size of  $U$  is

$$|U| = (\frac{3}{4}f_h + \frac{1}{4}f_a + \frac{3}{4}f_b + \frac{1}{4}f_c)s. \quad (10)$$

Now consider set  $L$ . Obviously, if Bob announces the result honestly, Alice will not detect it as a lie. Thus honest results will not present in  $L$ . We also showed above that set  $N$  will not contain any type  $b$  lie. Therefore, all the type  $b$  lies not included in  $U$  will present in  $L$ , so that the number is  $f_b s/4$ . Now suppose that  $|p''_i, q''_i\rangle_\beta = |0, 1\rangle_\beta$  is a type  $a$  lie and Alice has chosen  $q_i = 0$  to prepare the state  $|\alpha_i \otimes \beta_i\rangle$ . That is, Bob's actual result is  $|p'_i, q'_i\rangle_\beta = |0, 0\rangle_\beta$ . As we showed above, if Alice finds  $|x\rangle_\alpha$  in her measurement on  $\alpha_i$  then she detects  $|p''_i, q''_i\rangle_\beta = |0, 1\rangle_\beta$  as a lie. Eq. (7) shows that the case where Alice finds  $|x\rangle_\alpha$  while Bob finds  $|0, 0\rangle_\beta$  occurs with probability  $\cos^2 \theta_i$ . On the other hand, when  $|p''_i, q''_i\rangle_\beta = |1, 1\rangle_\beta$  is a type  $a$  lie and Alice has chosen  $q_i = 0$ , Eq. (8) shows that Alice will detect it (i.e., she finds  $|y\rangle_\alpha$  while Bob's actual result is  $|1, 0\rangle_\beta$ ) with probability  $\sin^2 \theta_i$ . In average, Bob's type  $a$  lies will be detected with probability  $(\cos^2 \theta_i + \sin^2 \theta_i)/2 = 1/2$ . Analyzing the  $q_i = 1$  case will also give the similar result. Consequently, the number of type  $a$  lies in set  $L$  is  $f_a s/2$ . Finally, when  $q_i = 0$ , if  $|p''_i, q''_i\rangle_\beta = |0, 1\rangle_\beta$  is a type  $c$  lie, Eq. (8) shows that Alice will detect it (i.e., she finds  $|x\rangle_\alpha$  while Bob's actual result is  $|1, 0\rangle_\beta$ ) with probability  $(\cos^2 \theta_i)/2$ . Else if  $|p''_i, q''_i\rangle_\beta = |1, 1\rangle_\beta$  is a type  $c$  lie, Eq. (7) shows that Alice will detect it (i.e., she finds  $|y\rangle_\alpha$  while Bob's actual result is  $|0, 0\rangle_\beta$ ) with probability  $(\sin^2 \theta_i)/2$ . In average, Bob's type  $c$  lies will be detected with probability  $((\cos^2 \theta_i)/2 + (\sin^2 \theta_i)/2)/2 = 1/4$ . So does the  $q_i = 1$  case. That is, the number of type  $c$  lies in set  $L$  is  $f_c s/4$ . Thus we know that the size of  $L$  is

$$|L| = (\frac{1}{2}f_a + \frac{1}{4}f_b + \frac{1}{4}f_c)s = l. \quad (11)$$

As a consequence, the size of  $N$  is

$$|N| = s - |U| - |L| = (\frac{1}{4}f_h + \frac{1}{4}f_a + \frac{1}{2}f_c)s, \quad (12)$$

and the size of  $M$  is

$$|M| = |L| + |N| = (\frac{1}{4}f_h + \frac{3}{4}f_a + \frac{1}{4}f_b + \frac{3}{4}f_c)s. \quad (13)$$

Combining with Eq. (9), we have

$$|M| = (\frac{1}{4} + \frac{f_a + f_c}{2})s. \quad (14)$$

In Fig.1 we summarized the above sizes and properties of these sets.

Thus we see that Alice can indeed detect  $l$  lies, i.e., achieve the same goal of the "semi-classical" algorithm, while only about  $|M| = [1/4 + (f_a + f_c)/2]s$  qubits are measured. This number is even less than that of algorithm II as long as  $f_a + f_c < 1/2$ .

#### D. Algorithm IV: alternative quantum algorithm with non-maximally entangled states

Similarly, we have the following algorithm.

*The state:* Alice prepares

$$|\alpha_i \otimes \beta_i\rangle = \cos \theta_i |x\rangle_\alpha \otimes |0, q_i\rangle_\beta + \sin \theta_i |y\rangle_\alpha \otimes |1, \neg q_i\rangle_\beta. \quad (15)$$

*Lie-detecting strategy:* After Bob announced  $|p''_i, q''_i\rangle_\beta$ , Alice divides the  $s$  pairs of entangled states into two subsets: Set  $M'$  includes all those satisfying either  $|p''_i, q''_i\rangle_\beta = |0, \neg q_i\rangle_\beta$  or  $|p''_i, q''_i\rangle_\beta = |1, q_i\rangle_\beta$ , and set  $U' \equiv S - M'$ . Again, she only measures each  $\alpha_i$  ( $i \in M'$ ) in the basis  $\{|x\rangle_\alpha, |y\rangle_\alpha\}$ , and sets  $p_i = 0$  ( $p_i = 1$ ) if she finds  $|x\rangle_\alpha$  ( $|y\rangle_\alpha$ ). Then she sets  $L' = \{i \in M' \wedge p_i = p''_i\}$  as the set of qubits for which she detected lies.

*Result:* With a similar analysis to that of algorithm III, it can be proven that set  $N' \equiv M' - L'$  does not contain any type  $c$  lie, and Alice needs to measure about  $|M'| = [1/4 + (f_a + f_b)/2]s$  qubits to detect  $l$  lies.

This number is also less than that of algorithm II as long as  $f_a + f_b < 1/2$ . But whether this algorithm is more efficient than algorithm III or not will depend on the comparison between  $f_b$  and  $f_c$ , which is determined by Bob. In the following we will use the case  $f_b > f_c$  as an example and build a QCF protocol upon algorithm III. But in fact algorithm IV can also be used to build a similar protocol in the case  $f_b < f_c$ .

### III. OUR PROTOCOL

Basing on the above lie-detecting problem and algorithm III, we build the following protocol.

*The QCF protocol (for generating a random bit  $c$ ):*

(1) Alice and Bob agree on a subset  $C$  of  $s$ -bit strings, i.e.,  $C \subset \{0, 1\}^s$ , which has the size  $|C| \sim 2^k$  ( $k < s$ ). Also, the elements of  $C$  (called codewords) should satisfy both of the following requirements:

- (i) The numbers of codewords having odd and even parity should both be non-trivial.
- (ii) The distance (i.e., the number of different bits) between any two codewords is not less than  $d$  ( $d < s$ ).

A binary linear  $(s, k, d)$ -code (or its selected subset, as we do not need it to meet all the requirements of linear classical error-correction code except the above two) generally fits the job.

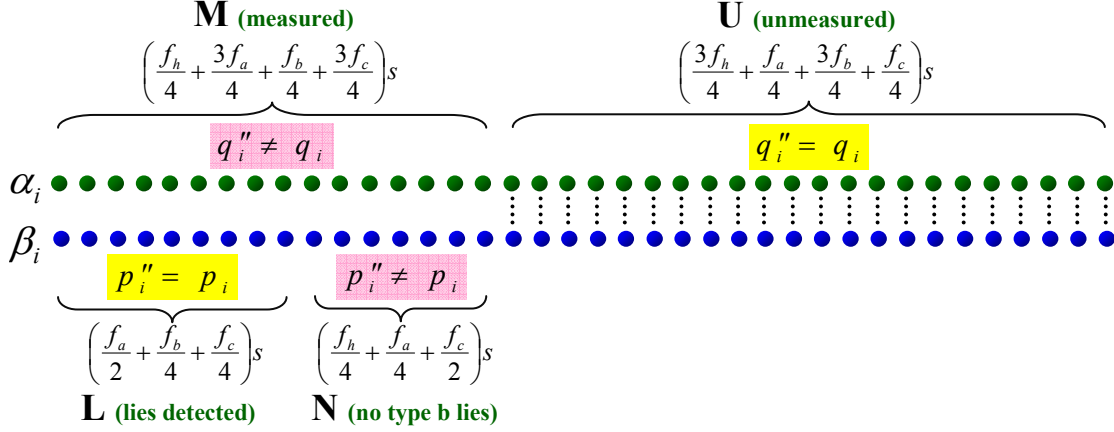


FIG. 1: Resultant subsets of the qubits after applying the lie-detecting algorithm III. Note that for illustration purposes, the qubits are regroupped in this diagram according to the subsets that they belong to, instead of following their original order indicated by the index  $i$ .

(2) Alice chooses a codeword  $q \equiv q_1 q_2 \dots q_s \in C$ , and prepares  $s$  pairs of quantum systems  $\alpha_i \otimes \beta_i$  ( $i \in S \equiv \{1, \dots, s\}$ ), each of which is in the state

$$|\alpha_i \otimes \beta_i\rangle = \frac{1}{\sqrt{2}}(|x\rangle_\alpha \otimes |0, q_i\rangle_\beta + |y\rangle_\alpha \otimes |1, q_i\rangle_\beta), \quad (16)$$

which is Eq. (5) with  $\theta_i$  fixed as  $\pi/4$ . She keeps each  $\alpha_i$  at her side, while sends  $\beta_i$  to Bob.

(3) Bob announces his “fake” measurement result  $|p''_i, q''_i\rangle_\beta$  for each  $\beta_i$ . Here the numbers of 0’s and 1’s in  $q'' \equiv q''_1 q''_2 \dots q''_s$  should both be non-trivial. Also, types  $a$ ,  $b$  and  $c$  lies should occur with frequencies  $f_a$ ,  $f_b$  and  $f_c$  that satisfy  $2d/s < f_a + f_c < 1/2$  and  $f_b > f_c$ .

(4) Alice checks that the numbers of 0’s and 1’s in  $q''$  are both non-trivial. Then she applies the lie-detecting strategy in algorithm III. Namely, for each  $\alpha_i$ , Alice leaves it unmeasured if  $q''_i = q_i$ . Else if  $q''_i \neq q_i$ , she measures  $\alpha_i$  in the basis  $\{|x\rangle_\alpha, |y\rangle_\alpha\}$ , and sets  $p_i = 0$  ( $p_i = 1$ ) if the result is  $|x\rangle_\alpha$  ( $|y\rangle_\alpha$ ). Then she divides set  $S$  into three subsets:

$$\begin{aligned} U &\equiv \{i \in S | q''_i = q_i\}, \\ L &\equiv \{i \in S | (q''_i \neq q_i) \wedge (p''_i = p_i)\}, \\ N &\equiv \{i \in S | (q''_i \neq q_i) \wedge (p''_i \neq p_i)\}. \end{aligned}$$

She also checks that the size of set  $M = L \cup N$  satisfies  $|M| > d + s/4$  (which guarantees that  $2d/s < f_a + f_c$  as can be seen from Eq. (14)).

(5) Alice announces set  $L$  to Bob.

(6) Bob checks that the size of  $L$  satisfies Eq. (11) approximately (A small amount of deviation is allowed, since the equation was merely a statistical estimation), and all  $|p''_i, q''_i\rangle_\beta$ ’s ( $i \in L$ ) are indeed lies. (No deviation allowed.)

(7) Bob announces a random bit  $f \in \{0, 1\}$ .

(8) Alice announces sets  $N$  and  $U$  to Bob, and sends him all  $\alpha_i$ ’s in set  $U$ .

(9) From the definitions of  $U$ ,  $L$ ,  $N$  Bob can deduce  $q$  from his own  $q''$ . Then he checks that:

(9.1)  $q$  is exactly a codeword of  $C$ .

(9.2) The sizes of  $N$  and  $U$  satisfy Eqs. (12) and (10) approximately. (A small amount of deviation is allowed.)

(9.3) Set  $N$  does not contain type  $b$  lies. (No deviation allowed.)

(9.4) He measures each  $\alpha_i$  in set  $U$  using any basis he prefers, and compares the result with his  $|p'_i, q'_i\rangle_\beta$  to verify that the state of  $\alpha_i \otimes \beta_i$  consists with both Eq. (16) and the  $q_i$  value that he deduced from Alice’s announced  $U$ .

(10) If the above security checks are passed, they take the coin-flip result as  $c = 0 \oplus f$  ( $c = 1 \oplus f$ ) if  $\sum_{i \in N} q_i$  is an even (odd) number.

#### IV. SECURITY PROOF

We can see that the main idea of the protocol is as follows. Bob requires Alice to solve the lie-detecting problem with the highest efficiency (i.e., the number of unmeasured  $\alpha_i$  should be maximized) as he will check the size of  $U$  and measures both  $\alpha_i$  and  $\beta_i$  for any  $i \in U$ . Thus Alice is forced to apply algorithm III when Bob chooses  $f_a + f_c < 1/2$  and  $f_b > f_c$  since other algorithms are less efficient in this case. As a consequence, the qubits are divided into subsets  $M$  and  $U$  by the comparison between  $q \equiv q_1 q_2 \dots q_s$  and  $q'' \equiv q''_1 q''_2 \dots q''_s$ , as shown in Fig.1. The coin-flip result  $c$  is essentially determined by the  $q_i$  values of the qubits felt into the subset  $N$ . Bob also generates a bit  $f$  in step (7) which serves as a flip of the final  $c$ , so that Alice’s biasing the states before this step will be in vain as different values of  $f$  could bias the final towards opposite directions. Therefore, to prove that the protocol is unconditionally secure, we only need to prove that Bob does not know Alice’s  $q$  (and therefore  $N$ ) be-

fore step (7), while Alice cannot change  $q$  (and therefore  $N$ ) after step (7).

Consider Bob's case first. Eq. (16) indicates that Bob's own operation that maximize his information on  $q_i$  is the one that distinguishes the reduced density matrix  $\rho_{q_i=0} \equiv (|0,0\rangle_\beta \langle 0,0| + |1,0\rangle_\beta \langle 1,0|)/2$  from  $\rho_{q_i=1} \equiv (|0,1\rangle_\beta \langle 0,1| + |1,1\rangle_\beta \langle 1,1|)/2$ . Calculations show that the maximal probability for Bob to learn  $q_i$  correctly is  $\cos^2(\pi/8) \simeq 0.8536$ . Also, from Fig.1 we know that type  $b$  lies and honest results stand a higher chance to be included in set  $U$  than in set  $N$ , while type  $c$  lies do the opposite. However, step (10) shows that  $c$  depends on the parity of the number of 1's in  $q_i$  ( $i \in N$ ). Bob's wrong guess on a single  $q_i$  value could completely change this parity. Therefore, Bob cannot rely on the probabilistic guess on  $q$  and  $N$ . If he cannot learn the complete string  $q$  precisely, the relationship between his guess on the parity of  $\sum_{i \in N} q_i$  and the actual value will be completely random, so that he cannot determine the correct  $f$  value to announce in step (7) that leads to his desired  $c$ .

The only exception is type  $b$  lies, as they never present in set  $N$ . Therefore once Bob tells a type  $b$  lie and Alice does not announce it as an element of set  $L$  in step (5), Bob knows with certainty that the corresponding  $i$  has to be included in set  $U$ , and he knows this  $q_i$  from his own  $q_i''$  as they are equal. So we must limit the total number of type  $b$  lies in the protocol. In step (4) Alice checks  $|M| > d + s/4$  which indicates  $2d/s < f_a + f_c$ . As Eqs. (12) and (10) show that the total number of type  $b$  lies in  $N \cup U$  is  $s_b \equiv 3f_b/4$ , we have

$$\begin{aligned} s_b &= (|N| + |U|) - (f_h + \frac{1}{2}f_a + \frac{3}{4}f_c)s \\ &\leq (|N| + |U|) - \frac{1}{2}(f_a + f_c)s \\ &< (|N| + |U|) - d. \end{aligned} \quad (17)$$

Also, since set  $L$  is announced in step (5), Bob knows every  $q_i$  ( $i \in L$ ). Therefore, the total number of  $q_i$ 's that Bob knows exactly is  $s_b + |L| < s - d$ . This result means that there are more than  $d$  bits of the  $s$ -bit string  $q$  remaining uncertain to Bob. From the definition of set  $C$  in step (1) we can see that knowing less than  $s - d$  bits of  $q$  is insufficient for Bob to determine which codeword it is. That is, the requirement  $|M| > d + s/4$  guarantees that Bob does not have enough information on  $q$  in step (7) to determine the parity of  $\sum_{i \in N} q_i$ . The operations after step (7) are merely the security checks against Alice's cheating. There is no more operation for Bob to affect the coin-flip outcome. Thus we see that the protocol is unconditionally secure against dishonest Bob.

Now consider Alice's cheating. Note that Bob's lying frequencies  $f_a$ ,  $f_b$  and  $f_c$  are never announced to Alice directly throughout the protocol. If Alice follows algorithm III honestly, the sizes of her announced sets  $U$ ,  $L$ ,  $N$  will meet Eqs. (10), (11) and (12) automatically. But if she does not prepare the initial states in the form of Eq. (5), while she may still find a set  $L$  with the proper

size, she has trivial probability to make a close guess on Bob's  $f_a$ ,  $f_b$ ,  $f_c$  (even though a small amount of statistical fluctuation is allowed) and announced sets  $U$ ,  $N$  with the correct size. Then she will be caught in step (9.2).

On the other hand, suppose that she prepared the states following Eq. (5), but after Bob announced  $|p_i'', q_i''\rangle_\beta$  in step (3), she does not like the relationship between  $q$  and  $q''$  as they will not lead to her desired  $c$  value. Then to alter  $c$ , she needs to announce some elements of  $N$  as the elements of  $U$  instead, or vice versa. But it is well-known that entanglement cannot be created locally. Once Alice measured  $\alpha_i$  in  $\alpha_i \otimes \beta_i$  (i.e.,  $i \in M$ ), it is no longer entangled with  $\beta_i$ . If she tries to claim that it is an element of the unmeasured set  $U$ , there is a non-trivial probability that in step (9.4) when Bob measures  $\alpha_i$  and compares it with his measurement on  $\beta_i$ , he will find a non-correlated result. For example, suppose that for a certain  $i$ , Bob has announced  $q_i'' = 0$ , and in step (8) Alice claims that  $i \in U$  which implies that  $q_i = q_i'' = 0$  and  $\alpha_i$  has not been measured. If Bob has measured the corresponding  $\beta_i$  and the actual result is  $|p_i', q_i'\rangle_\beta = |0,0\rangle_\beta$ , then from Eq. (7) he is expecting that  $\alpha_i$  has collapsed to the state  $(\cos\theta_i|x\rangle_\alpha + (\sin\theta_i/\sqrt{2})|y\rangle_\alpha)/n_{\alpha i}$ . Here  $n_{\alpha i}$  is the normalization constant, and we fixed  $\theta_i = \pi/4$  in the protocol. To check Alice's announcement, in step (9.4) Bob can measure this  $\alpha_i$  in the basis  $\{(\cos\theta_i|x\rangle_\alpha + (\sin\theta_i/\sqrt{2})|y\rangle_\alpha)/n_{\alpha i}, (\cos\theta_i|x\rangle_\alpha - (\sin\theta_i/\sqrt{2})|y\rangle_\alpha)/n_{\alpha i}\}$ , where  $(\cos\theta_i|x\rangle_\alpha + (\sin\theta_i/\sqrt{2})|y\rangle_\alpha)/n_{\alpha i}$  denotes the state orthogonal to  $(\cos\theta_i|x\rangle_\alpha + (\sin\theta_i/\sqrt{2})|y\rangle_\alpha)/n_{\alpha i}$ . If dishonest Alice already measured  $\alpha_i$  in the basis  $\{|x\rangle_\alpha, |y\rangle_\alpha\}$  or  $|\alpha_i \otimes \beta_i\rangle$  was not prepared as Eq. (16), then Bob has a non-trivial probability to find the measurement result as  $(\cos\theta_i|x\rangle_\alpha + (\sin\theta_i/\sqrt{2})|y\rangle_\alpha)/n_{\alpha i}$  and thus detect her cheating. Specially, if Bob has left a portion of  $\beta_i$ 's unmeasured before step (9.4) (note that we elaborated before algorithm I that this will not prevent Bob from controlling the values of  $f_a$ ,  $f_b$  and  $f_c$ ) and the current  $\beta_i$  happens to belong to this portion, then he can perform a collective measurement on  $\alpha_i \otimes \beta_i$  to see whether they can be projected to the state in Eq. (16).

Alice's announcing an element of  $U$  as that of  $N$  is also detectable. For  $\forall i \in U$ , Bob's announced  $|p_i'', q_i''\rangle_\beta$  satisfies  $q_i'' = q_i$ . Take  $q_i = 0$  for example. If Bob announced  $|p_i'', q_i''\rangle_\beta = |0,0\rangle_\beta$ , Eq. (7) shows that  $\alpha_i$  should collapse to  $(\cos\theta_i|x\rangle_\alpha + (\sin\theta_i/\sqrt{2})|y\rangle_\alpha)/n_{\alpha i}$  if  $|p_i'', q_i''\rangle_\beta$  is not a lie. So if Alice measures  $\alpha_i$  in the basis  $\{(\cos\theta_i|x\rangle_\alpha + (\sin\theta_i/\sqrt{2})|y\rangle_\alpha)/n_{\alpha i}, (\cos\theta_i|x\rangle_\alpha - (\sin\theta_i/\sqrt{2})|y\rangle_\alpha)/n_{\alpha i}\}$  and the result is  $(\cos\theta_i|x\rangle_\alpha + (\sin\theta_i/\sqrt{2})|y\rangle_\alpha)/n_{\alpha i}$ , she knows that Bob lies. But if Bob tells a type  $b$  lie, i.e., his actual result is  $|p_i', q_i'\rangle_\beta = |1,0\rangle_\beta$ , Eq. (8) shows that  $\alpha_i$  actually collapsed to  $((\cos\theta_i/\sqrt{2})|x\rangle_\alpha + \sin\theta_i|y\rangle_\alpha)/n'_{\alpha i}$ . It has a non-trivial probability to be detected as

$(\cos \theta_i |x\rangle_\alpha + (\sin \theta_i / \sqrt{2}) |y\rangle_\alpha) / n_{\alpha i}$  since they are not orthogonal. That is, type  $b$  lies cannot be detected with probability 100% when  $q_i'' = q_i$ , in contrast to the case  $q_i'' \neq q_i$ . Consequently, if Alice picks an element of  $U$  and wants to claim that it belongs to  $N$  instead, Eq. (10) shows that she stands probability  $3f_b/4$  to come across a type  $b$  lie, and she cannot always distinguish it even if she measures  $\alpha_i$ . As it was shown in algorithm III that  $N$  should not contain any type  $b$  lies, Alice's claiming  $i \in N$  will immediately be caught as cheating.

Thus it is shown that either Alice announces a single element of  $N$  as that of  $U$ , or vice versa, she stands a non-trivial probability  $\varepsilon$  to be detected. More importantly, while altering one single element of  $N$  and  $U$  (i.e., changing one single  $q_i$  value) could be sufficient for changing the parity of  $\sum_{i \in N} q_i$  and thus affect the final coin-flip outcome  $c$ , in our protocol the string  $q$  is required to be a codeword of  $C$ . As the minimal distance between codewords is  $d$ , changing one or few  $q_i$ 's of a codeword will result in  $q \notin C$  and be detected in step (9.1). Consequently, dishonest Alice has to alter at least  $d/2$  bits of  $q$  (suppose that she started with a  $q$  lying half way between two codewords) to cheat successfully. But then the total probability for escaping the detection will be at the order of magnitude of  $(1 - \varepsilon)^{d/2}$ . By choosing a very high  $s$  value in our protocol, the  $d$  value can also be increased, so that Alice's probability for successfully biasing the coin-flip result can be made arbitrarily close to zero. Thus our protocol is also unconditionally secure against dishonest Alice.

## V. RELATIONSHIP WITH THE NO-GO PROOFS

Currently there are many references on the no-go proofs of unconditionally secure QCF [5], [7–19]. Among them, Ref. [5] is a review on previous results, without supplying any new proof of its own. Ref. [8] provided the specific cheating strategy on the protocol in Ref. [20] only. Ref. [10] is merely the specific cheating strategy on a class of cheat-sensitive protocols proposed in Ref. [21]. The no-go proofs in Refs. [11, 19] are aimed at the QCF protocols built upon QBC. But we already elaborated in Refs. [27, 28] that the no-go proofs on unconditionally secure QBC is not sufficiently general to cover all protocols. Refs. [14, 15] studied a family of weak QCF protocols which are based on the  $n$ -coin-games defined in Ref. [15]. Ref. [18] studied the attack on two specific practical QCF protocols. That is, they are not supposed to be general.

The rest no-go proofs [7, 9, 12, 13, 16, 17] were claimed to apply to all kinds of QCF protocols. Thus our current result seems to conflict with them, and it is natural to question which points in their reasoning do not fit our protocol. As each of these proofs has its own approach, it is hard to answer to them all in one common sentence. Therefore, we will explain below how our protocol evades

these no-go proofs one by one.

Ref. [7]: This reference proved that ideal QCF (i.e., the bias  $\varepsilon$  equals to 0 precisely, instead of being arbitrarily close to 0) is impossible. As seen from the proof of its Theorem 2 in page 8, their approach is to argue that if an ideal QCF can be done in  $N$  rounds of communication, then it can be done in  $N - 1$  rounds. By repeating this induction, it seems that QCF can be done without any communication – which is an absurd result that can easily be disproved. But our protocol is non-ideal QCF. Furthermore, the  $N$  to  $N - 1$  rounds induction obviously cannot apply to our protocol. For example, step (9) of our protocol is a security check. If it is removed, then the protocol surely becomes an insecure QCF, which is not equivalent to the original one. Also, in step (8) Alice announces sets  $N$  and  $U$  which is necessary for Bob to calculate the coin-flip result  $c$ . If it is also removed, then Bob cannot get  $c$  so that it is not a complete QCF. Thus the induction approach does not work in our case.

Ref. [9]: This paper showed that to achieve a bias of at most  $\varepsilon$ , a QCF protocol must use at least  $\Omega(\log \log(1/\varepsilon))$  rounds of communication. According to the proof of its Lemma 12 on page 14, “to bias the coin towards 0, Alice just runs the honest protocol with her starting state being  $|\psi'_A\rangle$  instead of  $|\psi_A\rangle$ ”. Here  $|\psi_A\rangle$  denotes the initial state at Alice's side when the protocol is executed honestly,  $|\psi'_A\rangle$  is the remaining state when Alice applies  $M$  on  $|\psi_A\rangle$  and gets the coin-flip outcome 0, with  $M$  being the best measurement that distinguishes the two density matrices corresponding to the two outcomes 0 and 1, respectively. But in our protocol, such a measurement  $M$  (not to be confused with set  $M$ ) will be determined by Bob's choice of  $q''$  in step (3). Alice does not know it at the beginning of the protocol. Therefore, at this stage she cannot know what would be the form of  $|\psi'_A\rangle$ , so that this cheating strategy cannot be implemented.

Ref. [12]: This is known as the very first proof which lowered the bound to  $\varepsilon \geq 1/\sqrt{2} - 1/2$ . While this reference was widely cited, what can be found online is merely a scan of the slides. Thus the details of the proof is inaccessible. Fortunately, its result was reproduced in Ref. [13]. Therefore, we will analyze it based on [13] below.

Ref. [13]: According to its Definition 8, any QCF protocol is treated as a series of Alice's and Bob's unitary transformations  $U_{A,j}$ 's and  $U_{B,j}$ 's ( $1 \leq j \leq N$ ) on the Hilbert space, and their final measurements  $\Pi_{A,c}$  and  $\Pi_{B,c}$  for finding the coin-flip outcome  $c = 0, 1$  while the protocol does not abort. Its Lemmas 10 and 11 suggested that the optimal strategy of Bob trying to force outcome 1 is the solution to the primal semidefinite program (SDP) in its Eqs. (29)-(31), whose dual SDP is given by its Eqs. (32)-(34). Note that its Eqs. (29), (31), (33) and (34) all depend on Alice's  $U_{A,j}$ 's and  $\Pi_{A,1}$ . In our protocol, Alice's choices of the  $q$  value in step (2) play the role of  $U_{A,j}$ 's, which is unknown to Bob before he needs to announce  $q''$  in step (3) and  $f$  in step (7). Since the coin-flip result  $c$  is determined by  $f$  and the comparison between  $q$  and  $q''$ , once Bob announced  $q''$  and  $f$ ,  $c$  is

fixed so that he can no longer alter it. Therefore, though the optimal SDP that leads to the cheating probability  $p_{*1} \geq 1/\sqrt{2}$  (i.e., the Kitaev's bound  $\varepsilon \geq 1/\sqrt{2} - 1/2$ ) exists, Bob does not have enough information on  $U_{A,j}$ 's to compute it before the coin-flip result is generated. The same analysis also applies to dishonest Alice. This result is very similar to the case of Ref. [16] below, where the optimal cheating strategy exists but the cheater cannot reach it.

Ref. [16]: This proof recovered Kitaev's bound  $\varepsilon \geq 1/\sqrt{2} - 1/2$  for strong QCF [12] with a different presentation. In brief, as shown in its page 8, suppose  $\{A_0, A_1, A_{\text{abort}}\}$  is honest-Alice's strategy and  $\{B_0, B_1, B_{\text{abort}}\}$  is honest-Bob's co-strategy in a QCF protocol, corresponding to the outcomes 0, 1 and *abort*, respectively. The definition of QCF implies

$$\frac{1}{2} = \langle A_0, B_0 \rangle = \langle A_1, B_1 \rangle. \quad (18)$$

Let  $p$  be the maximum probability that a cheating Bob can force honest-Alice to output a fixed  $c \in \{0, 1\}$ . Then its Theorem 9 implies that there must exist a strategy  $Q$  for Alice such that  $A_c \leq pQ$ . If a cheating Alice plays this strategy  $Q$ , then honest-Bob outputs  $c$  with probability

$$\langle Q, B_c \rangle \geq \frac{1}{p} \langle A_c, B_c \rangle = \frac{1}{2p}. \quad (19)$$

Given that

$$\max \left\{ p, \frac{1}{2p} \right\} \geq \frac{1}{\sqrt{2}} \quad (20)$$

for all  $p > 0$ , either honest-Alice or honest-Bob can be convinced to output  $c$  with probability at least  $1/\sqrt{2}$ , so that the bias satisfies  $\varepsilon \geq 1/\sqrt{2} - 1/2$ .

Now consider our protocol. First let us suppose that Alice is honest, then the quantum states Bob received are his halves of the entangled states faithfully prepared as Eq. (16). As a result, Bob cannot learn the exact values of Alice's  $q_i$ 's from his own measurement, since any value of  $|p'_i, q'_i\rangle_\beta$  is possible no matter  $q_i = 0$  or  $q_i = 1$ . Also,  $c$  is determined by the parity of  $\sum_{i \in N} q_i$ , which depends sensitively on the value of every single  $q_i$ . Consequently, making a probabilistic guess of  $q_i$  is useless for Bob to alter  $c$ . Therefore, according to the above description of Ref. [16], our protocol has  $p = 1/2$ . Then Eq. (19) indicates that there is a cheating strategy  $Q$  which can maximize  $\langle Q, B_c \rangle$  to 1, which is Alice's probability for forcing the output.

However, the question is whether Alice knows this  $Q$ . Note that the task of  $Q$  is not merely producing Alice's desired  $c$  value, but also bringing her through the security checks successfully. In our protocol, to obtain the

output  $c$ , Bob's strategies are not limited to one single  $B_c$ . Even if he always announced the same values of  $|p'_i, q'_i\rangle_\beta$ 's and  $f$  in steps (3) and (7), his choice on the actual measurement bases  $p' \equiv p'_1 p'_2 \dots p'_s$  can be different, resulting in different locations and types of his lies. Also, in step (9.4) Bob has the freedom on choosing the measurement bases  $p'^{(\alpha)} \equiv \bigotimes_{i \in U} p'_i{}^{(\alpha)}$  for checking  $\alpha_i$ 's in set  $U$ . That is, even for the same  $|p'_i, q'_i\rangle_\beta$ 's and  $f$  he still has many different strategies, each of which is corresponding to a different choice of  $p'$  and  $p'^{(\alpha)}$ , so we can denote it as  $B_c(p', p'^{(\alpha)})$ . While  $p'$  and  $p'^{(\alpha)}$  do not affect  $c$ , they determine what kinds of Alice's states can pass Bob's security checks. Therefore, Alice strategy  $Q$  has to match Bob's  $B_c(p', p'^{(\alpha)})$ , so we can denote it as  $Q(B_c(p', p'^{(\alpha)}))$ .

As each  $p'_i$  has two possible values, there are totally  $2^s$  different choices for  $p'$ , leading to at least  $2^s$  different strategies  $B_c(p', p'^{(\alpha)})$ . The choices for  $p'^{(\alpha)}$  make the number of  $B_c(p', p'^{(\alpha)})$  even higher. Now, recall that in our protocol  $p'$  and  $p'^{(\alpha)}$  are never required to be announced to Alice. Consequently, although the optimal cheating strategy  $Q(B_c(p', p'^{(\alpha)}))$  that maximize  $\langle Q(B_c(p', p'^{(\alpha)})), B_c(p', p'^{(\alpha)}) \rangle$  could exist, Alice has more than  $2^s$  different  $Q(B_c(p', p'^{(\alpha)}))$  to choose from but she does not know which is the optimal one since she does not know  $p'$  and  $p'^{(\alpha)}$ . Thus she only has less than probability  $1/2^s$  to find the optimal  $Q(B_c(p', p'^{(\alpha)}))$  and reach the maximal bias described in Eq. (19). That is why the proof in Ref. [16] does not apply to our protocol.

Ref. [17]: As can be seen from its section 4.2, its no-go proof on strong QCF is based on that of [13]. Thus it does not apply to our protocol for the same reason that we elaborated above. It is also worth pinpointing out a problem which may hurt the generality of this proof, though it does not related directly with our protocol. That is, footnote 4 on its page 3 is incorrect. It said that "the players can always add a final round to check if they have the same value and output the dummy symbol if the values differ". But if there is a secure QCF protocol without this round, then adding such a round will make it insecure, because a dishonest party will be able to bias the output by always announcing the value he preferred as what he obtained from the protocol, regardless what is the actual output value. Whenever his preferred value differs from the actual one, the protocol will abort so he has nothing to lose. Therefore, dummy output in QCF should not be allowed to be generated in the way described in this footnote.

In summary, our protocol has features that are not covered in all the above no-go proofs. Thus it can break the constraint of existing security bounds on QCF.

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